Valuations and S-units

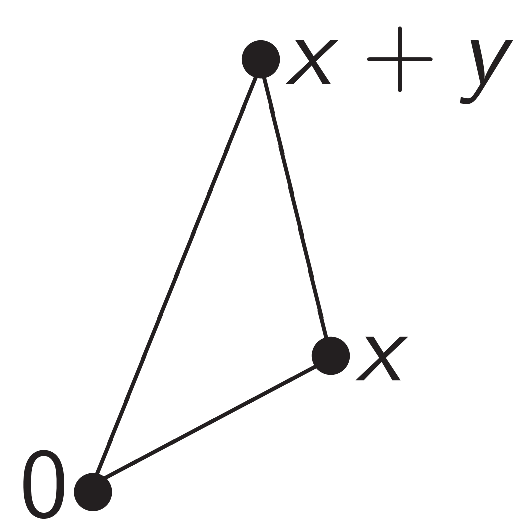
D. J. Bernstein

University of Illinois at Chicago;
Ruhr University Bochum

$\mathbf{R}$ = field of real numbers.
$\mathbf{C}$ = field of complex numbers.

The function $x \mapsto |x|$
from $\mathbf{C}$ to $\mathbf{R}$ is a **valuation on C**:

- $|0| = 0$.
- $x \neq 0 \Rightarrow |x| > 0$.
- $|xy| = |x||y|$.
- $|x + y| \leq |x| + |y|$.



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e.g. $x \mapsto \sqrt{|x|}$ is a valuation.
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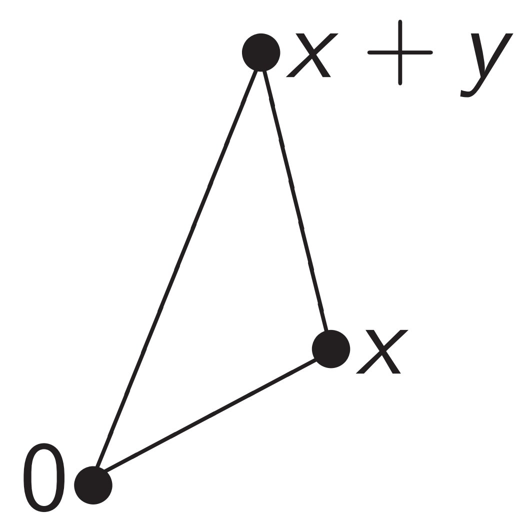

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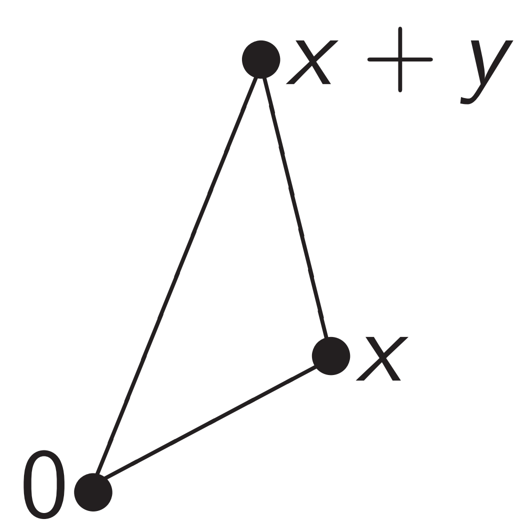

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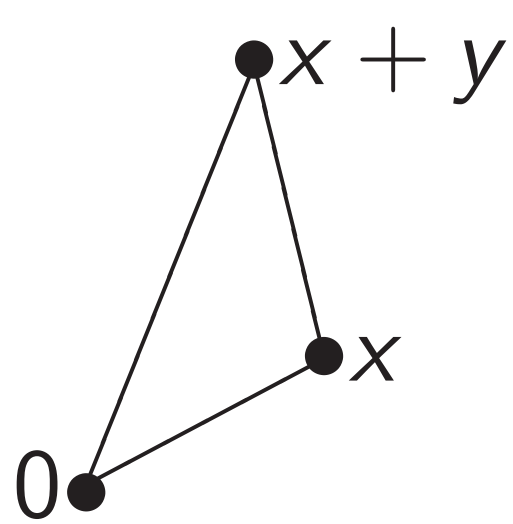

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positive powers of each other.
They have the same unit disks:
they map the same inputs to $\mathbf{R}_{\leq 1}$.

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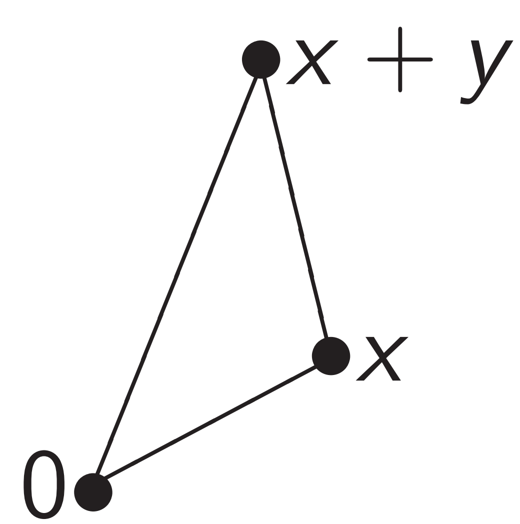

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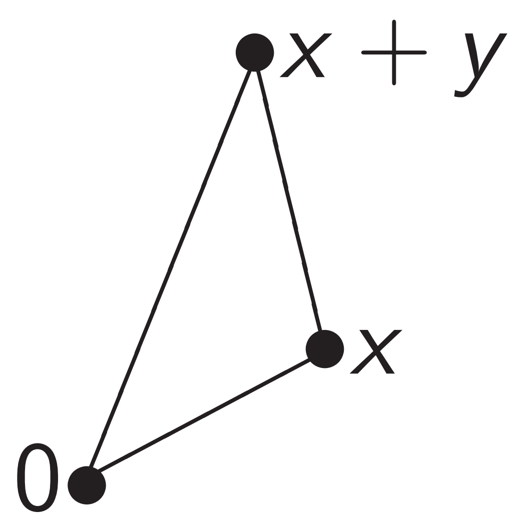

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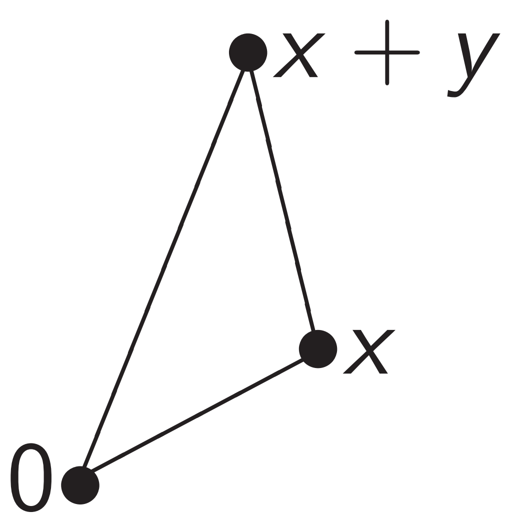

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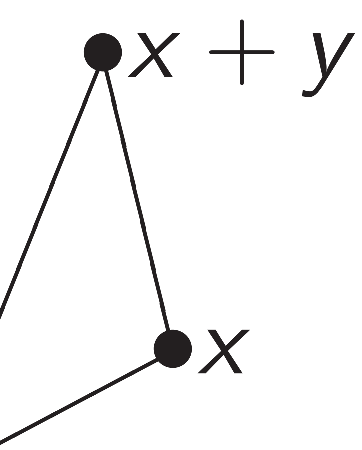

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A nonequivalent nontrivial
valuation on $\mathbf{Q}$: define $|0|_3 = 0$,
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For $x \in$

$|x|_p = p$

x	$\lvert x \rvert$
⋮	
−2	2
−1	1
0	0
1	1
2	2
3	3
4	4
5	5
6	6
⋮	

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$y| \le \sqrt{|x|} + \sqrt{|y|}$.

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$|x|_p = p^{-e_p}$ if $x =$

x	$\|x\|_\infty$	$\|x\|_2$	$\|x\|$
$\vdots$			
-2	2	1/2	1
-1	1	1	1
0	0	0	0
1	1	1	1
2	2	1/2	1
3	3	1	1/
4	4	1/4	1
5	5	1	1
6	6	1/2	1/
$\vdots$			

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x	$\|x\|_\infty$	$\|x\|_2$	$\|x\|_3$	$\|x\|_5$	$\ldots$
$\vdots$					
-2	2	1/2	1	1	$\ldots$
-1	1	1	1	1	$\ldots$
0	0	0	0	0	$\ldots$
1	1	1	1	1	$\ldots$
2	2	1/2	1	1	$\ldots$
3	3	1	1/3	1	$\ldots$
4	4	1/4	1	1	$\ldots$
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$\vdots$		[don't forget $x = 2/$			

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x	$\lvert x\rvert_\infty$	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	...	product
$\vdots$						
-2	2	1/2	1	1	...	1
-1	1	1	1	1	...	1
0	0	0	0	0	...	0
1	1	1	1	1	...	1
2	2	1/2	1	1	...	1
3	3	1	1/3	1	...	1
4	4	1/4	1	1	...	1
5	5	1	1	1/5	...	1
6	6	1/2	1/3	1	...	1
$\vdots$						

[don't forget $x = 2/3$ etc.]

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x	$\lvert x\rvert_\infty$	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	$\ldots$	product
$\vdots$						
-2	2	1/2	1	1	$\ldots$	1
-1	1	1	1	1	$\ldots$	1
0	0	0	0	0	$\ldots$	0
1	1	1	1	1	$\ldots$	1
2	2	1/2	1	1	$\ldots$	1
3	3	1	1/3	1	$\ldots$	1
4	4	1/4	1	1	$\ldots$	1
5	5	1	1	1/5	$\ldots$	1
6	6	1/2	1/3	1	$\ldots$	1
$\vdots$						

[don't forget $x = 2/3$ etc.]

Infinite-c

(log $|x|_\infty$

log $\lvert x\rvert_\infty$
$\vdots$
log 2
0
[skip $x =$
0
log 2
log 3
log 4
log 5
log 6
$\vdots$

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x	$\lvert x\rvert_\infty$	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	$\ldots$	product
$\vdots$						
-2	2	1/2	1	1	$\ldots$	1
-1	1	1	1	1	$\ldots$	1
0	0	0	0	0	$\ldots$	0
1	1	1	1	1	$\ldots$	1
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6	6	1/2	1/3	1	$\ldots$	1
$\vdots$						

[don't forget $x = 2/3$ etc.]

Infinite-dimensiona

$(\log |x|_\infty, \log |x|_2,$

$\log \lvert x\rvert_\infty$	$\log \lvert x\rvert_2$	lo
$\vdots$		
$\log 2$	$-\log 2$	0
0	0	0
[skip $x = 0$: $\log 0$		
0	0	0
$\log 2$	$-\log 2$	0
$\log 3$	0	$-$
$\log 4$	$-\log 4$	0
$\log 5$	0	0
$\log 6$	$-\log 2$	$-$
$\vdots$		

[again don't

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$^{e_5} \dots.$

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$\vdots$						
-2	2	1/2	1	1	$\dots$	1
-1	1	1	1	1	$\dots$	1
0	0	0	0	0	$\dots$	0
1	1	1	1	1	$\dots$	1
2	2	1/2	1	1	$\dots$	1
3	3	1	1/3	1	$\dots$	1
4	4	1/4	1	1	$\dots$	1
5	5	1	1	1/5	$\dots$	1
6	6	1/2	1/3	1	$\dots$	1
$\vdots$	[don't forget $x = 2/3$ etc.]					

Infinite-dimensional lattice o
$(\log |x|_\infty, \log |x|_2, \log |x|_3, \dots$

$\log \lvert x\rvert_\infty$	$\log \lvert x\rvert_2$	$\log \lvert x\rvert_3$	log
$\vdots$			
$\log 2$	$-\log 2$	0	0
0	0	0	0
[skip $x = 0$: log 0 not define			
0	0	0	0
$\log 2$	$-\log 2$	0	0
$\log 3$	0	$-\log 3$	0
$\log 4$	$-\log 4$	0	0
$\log 5$	0	0	$-\mathrm{l}$
$\log 6$	$-\log 2$	$-\log 3$	0
$\vdots$	[again don't forget 2/3		

For $x \in \mathbf{Q}$, define $|x|_\infty = |x|$;
$|x|_p = p^{-e_p}$ if $x = \pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots$.

x	$\lvert x\rvert_\infty$	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	$\ldots$	product
$\vdots$						
-2	2	1/2	1	1	$\ldots$	1
-1	1	1	1	1	$\ldots$	1
0	0	0	0	0	$\ldots$	0
1	1	1	1	1	$\ldots$	1
2	2	1/2	1	1	$\ldots$	1
3	3	1	1/3	1	$\ldots$	1
4	4	1/4	1	1	$\ldots$	1
5	5	1	1	1/5	$\ldots$	1
6	6	1/2	1/3	1	$\ldots$	1
$\vdots$						

[don't forget $x = 2/3$ etc.]

Infinite-dimensional lattice of
$(\log |x|_\infty, \log |x|_2, \log |x|_3, \ldots)$:

$\log \lvert x\rvert_\infty$	$\log \lvert x\rvert_2$	$\log \lvert x\rvert_3$	$\log \lvert x\rvert_5$	$\ldots$
$\vdots$				
$\log 2$	$-\log 2$	0	0	$\ldots$
0	0	0	0	$\ldots$
[skip $x = 0$: log 0 not defined]				
0	0	0	0	$\ldots$
$\log 2$	$-\log 2$	0	0	$\ldots$
$\log 3$	0	$-\log 3$	0	$\ldots$
$\log 4$	$-\log 4$	0	0	$\ldots$
$\log 5$	0	0	$-\log 5$	$\ldots$
$\log 6$	$-\log 2$	$-\log 3$	0	$\ldots$
$\vdots$				

[again don't forget 2/3 etc.]

Q, define $|x|_\infty = |x|$;

$^{-e_p}$ if $x = \pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots$.

∞	$|x|_2$	$|x|_3$	$|x|_5$	$\ldots$	product
	$1/2$	1	1	$\ldots$	1
	1	1	1	$\ldots$	1
	0	0	0	$\ldots$	0
	1	1	1	$\ldots$	1
	$1/2$	1	1	$\ldots$	1
	1	$1/3$	1	$\ldots$	1
	$1/4$	1	1	$\ldots$	1
	1	1	$1/5$	$\ldots$	1
	$1/2$	$1/3$	1	$\ldots$	1

[don't forget $x = 2/3$ etc.]

Infinite-dimensional lattice of $(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$:

$\log|x|_\infty$	$\log|x|_2$	$\log|x|_3$	$\log|x|_5$	$\ldots$
$\vdots$				
$\log 2$	$-\log 2$	0	0	$\ldots$
0	0	0	0	$\ldots$
[skip $x = 0$: log 0 not defined]				
0	0	0	0	$\ldots$
$\log 2$	$-\log 2$	0	0	$\ldots$
$\log 3$	0	$-\log 3$	0	$\ldots$
$\log 4$	$-\log 4$	0	0	$\ldots$
$\log 5$	0	0	$-\log 5$	$\ldots$
$\log 6$	$-\log 2$	$-\log 3$	0	$\ldots$
$\vdots$				

[again don't forget 2/3 etc.]

This latt

$(\log|x|_\infty$

$(\log 2, -$

$(\log 3, 0,$

$(\log 5, 0,$

$(\log 7, 0,$

$\cdots$ wher

$\mathbf{Z} = \{\ldots$

$|x|_\infty = |x|$;

$= \pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots$.

$_3$	$\lvert x\rvert_5$	...	product
	1	...	1
	1	...	1
	0	...	0
	1	...	1
	1	...	1
3	1	...	1
	1	...	1
	1/5	...	1
3	1	...	1

get $x = 2/3$ etc.]

Infinite-dimensional lattice of $(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$:

$\log\lvert x\rvert_\infty$	$\log\lvert x\rvert_2$	$\log\lvert x\rvert_3$	$\log\lvert x\rvert_5$	...
⋮				
log 2	− log 2	0	0	...
0	0	0	0	...
[skip $x = 0$: log 0 not defined]				
0	0	0	0	...
log 2	− log 2	0	0	...
log 3	0	− log 3	0	...
log 4	− log 4	0	0	...
log 5	0	0	− log 5	...
log 6	− log 2	− log 3	0	...
⋮				

[again don't forget 2/3 etc.]

This lattice, the se

$(\log|x|_\infty, \log|x|_2,$

$(\log 2, -\log 2, 0, 0$

$(\log 3, 0, -\log 3, 0$

$(\log 5, 0, 0, -\log 5$

$(\log 7, 0, 0, 0, -\log$

$\cdots$ where

$\mathbf{Z} = \{\ldots, -2, -1,$

$|$;

$5^{e_5}\ldots$.

product
⋮
1
1
0
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3 etc.]

Infinite-dimensional lattice of $(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$:

$\log\|x\|_\infty$	$\log\|x\|_2$	$\log\|x\|_3$	$\log\|x\|_5$	$\ldots$
⋮				
$\log 2$	$-\log 2$	0	0	$\ldots$
0	0	0	0	$\ldots$
[skip $x = 0$: log 0 not defined]				
0	0	0	0	$\ldots$
$\log 2$	$-\log 2$	0	0	$\ldots$
$\log 3$	0	$-\log 3$	0	$\ldots$
$\log 4$	$-\log 4$	0	0	$\ldots$
$\log 5$	0	0	$-\log 5$	$\ldots$
$\log 6$	$-\log 2$	$-\log 3$	0	$\ldots$
⋮	[again don't forget 2/3 etc.]			

This lattice, the set of vecto

$(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots$

$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z} +$

$(\log 3, 0, -\log 3, 0, 0, \ldots)\mathbf{Z} +$

$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z} +$

$(\log 7, 0, 0, 0, -\log 7, \ldots)\mathbf{Z} +$

$\cdots$ where

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$\log 2$	$-\log 2$	0	0	$\ldots$
$\log 3$	0	$-\log 3$	0	$\ldots$
$\log 4$	$-\log 4$	0	0	$\ldots$
$\log 5$	0	0	$-\log 5$	$\ldots$
$\log 6$	$-\log 2$	$-\log 3$	0	$\ldots$
$\vdots$	[again don't forget 2/3 etc.]			

This lattice, the set of vectors $(\log |x|_\infty, \log |x|_2, \log |x|_3, \ldots)$, is

$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z} +$

$(\log 3, 0, -\log 3, 0, 0, \ldots)\mathbf{Z} +$

$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z} +$

$(\log 7, 0, 0, 0, -\log 7, \ldots)\mathbf{Z} +$

$\cdots$ where

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$(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$:

$\log\|x\|_\infty$	$\log\|x\|_2$	$\log\|x\|_3$	$\log\|x\|_5$	$\ldots$
$\vdots$				
$\log 2$	$-\log 2$	0	0	$\ldots$
0	0	0	0	$\ldots$
[skip $x = 0$: log 0 not defined]				
0	0	0	0	$\ldots$
$\log 2$	$-\log 2$	0	0	$\ldots$
$\log 3$	0	$-\log 3$	0	$\ldots$
$\log 4$	$-\log 4$	0	0	$\ldots$
$\log 5$	0	0	$-\log 5$	$\ldots$
$\log 6$	$-\log 2$	$-\log 3$	0	$\ldots$
$\vdots$	[again don't forget 2/3 etc.]			

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$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z} +$

$(\log 7, 0, 0, 0, -\log 7, \ldots)\mathbf{Z} +$

$\cdots$ where

$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$x = \pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots$ maps to

$(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots) =$

$(\log 2, -\log 2, 0, 0, 0, \ldots)e_2 +$

$(\log 3, 0, -\log 3, 0, 0, \ldots)e_3 +$

$(\log 5, 0, 0, -\log 5, 0, \ldots)e_5 +$

$(\log 7, 0, 0, 0, -\log 7, \ldots)e_7 +$

$\ldots$

dimensional lattice of

∞, $\log|x|_2, \log|x|_3, \ldots)$:

$\log\lvert x\rvert_2$	$\log\lvert x\rvert_3$	$\log\lvert x\rvert_5$	$\ldots$
$-\log 2$	0	0	$\ldots$
0	0	0	$\ldots$

= 0: log 0 not defined]

0	0	0	$\ldots$
$-\log 2$	0	0	$\ldots$
0	$-\log 3$	0	$\ldots$
$-\log 4$	0	0	$\ldots$
0	0	$-\log 5$	$\ldots$
$-\log 2$	$-\log 3$	0	$\ldots$

gain don't forget 2/3 etc.]

This lattice, the set of vectors $(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$, is

$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z} +$
$(\log 3, 0, -\log 3, 0, 0, \ldots)\mathbf{Z} +$
$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z} +$
$(\log 7, 0, 0, 0, -\log 7, \ldots)\mathbf{Z} +$
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$(\log 5, 0, 0, -\log 5, 0, \ldots)e_5 +$
$(\log 7, 0, 0, 0, -\log 7, \ldots)e_7 +$
$\ldots$

Can divi
obtain a
$\text{ord}_p(\pm 2$

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- leavin
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- want
  $\prod_v |x|$
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al lattice of
log $|x|_3, \ldots$):

og $|x|_3$	log $|x|_5$	...
)	0	...
)	0	...
not defined]		
)	0	...
)	0	...
$-\log 3$	0	...
)	0	...
)	$-\log 5$	...
$-\log 3$	0	...

forget 2/3 etc.]

This lattice, the set of vectors
$(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$, is
$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z} +$
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$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z} +$
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$(\log 5, 0, 0, -\log 5, 0, \ldots)e_5 +$
$(\log 7, 0, 0, 0, -\log 7, \ldots)e_7 +$
$\ldots$

Can divide $\log|x|_p$
obtain an integer
$\mathrm{ord}_p(\pm 2^{e_2} 3^{e_3} 5^{e_5} \cdot$

Number theorists
$\log p$ weight for m

- leaving out the v
  produce infinitely
  log vectors (e.g.
- want "the produ
  $\prod_v |x|_v = 1$; $\sum$
- this particular p
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Can divide $\log|x|_p$ by $\log p$ t
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$\operatorname{ord}_p(\pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots) = e_p$.

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Number theorists include the $\log p$ weight for many reasons:

- leaving out the weight would produce infinitely many short log vectors (e.g., length <2);
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et of vectors

$\log|x|_3, \ldots)$, is

$, 0, \ldots)\mathbf{Z} +$

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$, 0, \ldots)\mathbf{Z} +$

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$\cdot$ maps to

$\log|x|_3, \ldots) =$

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Say $S \subseteq \{\infty, 2, 3,$

Typical case: $p \in$

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Say $S \subseteq \{\infty, 2, 3, 5, \ldots\}$, ∞

Typical case: $p \in S \Leftrightarrow p \leq$

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Say $S \subseteq \{\infty, 2, 3, 5, \ldots\}$, $\infty \in S$.

Typical case: $p \in S \Leftrightarrow p \leq 37$.

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$x \in \mathbf{Q}$ is called an **S-integer** if $|x|_p \leq 1$ for each $p \notin S$.

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$\{S\text{-integers}\}$ is a subring of $\mathbf{Q}$: closed under mult since $\mathbf{R}_{\leq 1}$ is; closed under addition since $|x+y|_p \leq \max\{|x|_p, |y|_p\}$.

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For any commutative ring R: R^* means $\{u \in R : uR = R\}$.

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$x \in \mathbf{Q}^*$ is called an ***S*-unit** if $|x|_p = 1$ for each $p \notin S$. $\{S\text{-units}\} = \{S\text{-integers}\}^*$.

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$p = 2 + i$:	$\|p\|^2 = 5$.
$p = 2 - i$:	$\|p\|^2 = 5$.
$p = 7$:	$\|p\|^2 = 49$.
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$p = 3$: $|p|^2 = 9$.

$p = 2 + i$: $|p|^2 = 5$.

$p = 2 - i$: $|p|^2 = 5$.

$p = 7$: $|p|^2 = 49$.

$p = 11$: $|p|^2 = 121$.

$p = 3 + 2i$: $|p|^2 = 13$.

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e.g. $\zeta_8 = (1+i)/\sqrt{2}$; $\zeta_8^2 = \zeta_4 = i$.
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Fact: Each $x \in \mathbf{Q}(\zeta_8)^*$ factors uniquely as $r u^{e_u} \prod_{p\in P} p^{e_p}$ where $r \in \{1, \zeta_8, \ldots, \zeta_8^7\}$;
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$u = 1+\zeta_8+\zeta_8^2$; $e_u \in \mathbf{Z}$; $e_p \in \mathbf{Z}$.

Variant appearing in literature:
Split $|x|_\infty$ into two copies of $|x|$.
Gives slightly different lattice:

$(0.5\log 2, 0.5\log 2, -\log 2, 0, 0, 0, \ldots)$
$(0.5\log 9, 0.5\log 9, 0, -\log 9, 0, 0, \ldots)$
$(0.5\log 5, 0.5\log 5, 0, 0, -\log 5, 0, \ldots)$
$(0.5\log 5, 0.5\log 5, 0, 0, 0, -\log 5, \ldots)$
$\vdots$

Minor advantages: e.g., some definitions of the lattice become slightly more concise.

But now have redundant columns, each column deviating from the probability interpretation.

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Exercise: u valuation is trivial.

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⋮

n power ∞_1:

n power ∞_3:

where

$\zeta_8^2 + a_3\zeta_8^3)$

$_8^6 + a_3\zeta_8^9$.

$= \sigma_3(x)\sigma_3(y)$.

e inequivalent:

$\sqrt{2} > 1$,

$+ \sqrt{2}) < 1$.

n for $p \in P$:

sing prime

$_{\infty_1}|p|_{\infty_3}$.

$\{\infty_1, \infty_3\}$-integers:
$\mathbf{Z}[\zeta_8] = \mathbf{Z} + \mathbf{Z}\zeta_8 + \mathbf{Z}\zeta_8^2 + \mathbf{Z}\zeta_8^3$.

$\{\infty_1, \infty_3\}$-units: $\zeta_8^{\{0,\ldots,7\}} u^{\mathbf{Z}}$.

$\{\infty_1, \infty_3\}$-unit lattice:
$(1.76\ldots, -1.76\ldots, 0, \ldots)\mathbf{Z}$.

Again increase S for more S-units.

$\{\infty_1, \infty_3, 1 + \zeta_8\}$-units:
$\zeta_8^{\{0,\ldots,7\}} u^{\mathbf{Z}} (1 + \zeta_8)^{\mathbf{Z}}$.

$\{\infty_1, \infty_3, 1 + \zeta_8\}$-unit lattice:
$(1.76\ldots, -1.76\ldots, 0, \ldots)\mathbf{Z} +$
$(1.22\ldots, -0.53\ldots, -0.69\ldots, \ldots)\mathbf{Z}$.

Reasonably short b
for the infinite-dim
lattice of $\mathbf{Q}(\zeta_8)^*$ l
shown truncated a

1.76	−1.76	0
1.22	−0.53	−0.69
1.09	1.09	0
1.09	1.09	0
⋮		

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$\circ_3$:

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P:

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Reasonably short basis
for the infinite-dimensional
lattice of $\mathbf{Q}(\zeta_8)^*$ logs,
shown truncated after 2 digi

1.76	−1.76	0	0	(
1.22	−0.53	−0.69	0	(
1.09	1.09	0	−2.19	(
1.09	1.09	0	0	−2

⋮

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1.76	−1.76	0	0	0	...
1.22	−0.53	−0.69	0	0	...
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1.09	1.09	0	0	−2.19	...
⋮					

Diagonal after 2 columns.
Compare to the lattice bases for
$\mathbf{Q}$, $\mathbf{Q}(i)$: diagonal after 1 column.

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1.76	−1.76	0	0	0	...
1.22	−0.53	−0.69	0	0	...
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1.09	1.09	0	0	−2.19	...
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Diagonal after 2 columns.
Compare to the lattice bases for
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Exercise: Find shorter basis.

$_3\}$-integers:

$\mathbf{Z} + \mathbf{Z}\zeta_8 + \mathbf{Z}\zeta_8^2 + \mathbf{Z}\zeta_8^3$.

$_3\}$-units: $\zeta_8^{\{0,\ldots,7\}} u^{\mathbf{Z}}$.

$_3\}$-unit lattice:

$, -1.76\ldots, 0, \ldots)\mathbf{Z}$.

crease S for more S-units.

$_3, 1+\zeta_8\}$-units:

$u^{\mathbf{Z}}(1+\zeta_8)^{\mathbf{Z}}$.

$_3, 1+\zeta_8\}$-unit lattice:

$, -1.76\ldots, 0, \ldots)\mathbf{Z} +$

$, -0.53\ldots, -0.69\ldots, \ldots)\mathbf{Z}$.

Reasonably short basis
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shown truncated after 2 digits:

1.76	−1.76	0	0	0	...
1.22	−0.53	−0.69	0	0	...
1.09	1.09	0	−2.19	0	...
1.09	1.09	0	0	−2.19	...
⋮					

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The 16t

$\zeta_{16} =$ ex

$\mathbf{Q}(\zeta_{16}) =$

$+ \mathbf{Q}\zeta$

s:

$- \mathbf{Z}\zeta_8^2 + \mathbf{Z}\zeta_8^3$.

$\zeta_8^{\{0,\ldots,7\}} u^{\mathbf{Z}}$.

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$., 0, \ldots)\mathbf{Z}$.

for more S-units.

-units:

$)^{\mathbf{Z}}$.

-unit lattice:

$., 0, \ldots)\mathbf{Z} +$

$., -0.69\ldots, \ldots)\mathbf{Z}$.

Reasonably short basis
for the infinite-dimensional
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1.76	−1.76	0	0	0	...
1.22	−0.53	−0.69	0	0	...
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The 16th cycloton

$\zeta_{16} = \exp(2\pi i/16$

$\mathbf{Q}(\zeta_{16}) = \mathbf{Q} + \mathbf{Q}\zeta_1$

$+ \mathbf{Q}\zeta_{16}^4 + \mathbf{Q}\zeta_{16}^5$

ζ_8^3.

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$\ldots,\ldots)\mathbf{Z}$.

Reasonably short basis
for the infinite-dimensional
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1.76	−1.76	0	0	0	...
1.22	−0.53	−0.69	0	0	...
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The 16th cyclotomic field

$\zeta_{16} = \exp(2\pi i/16)$ so $\zeta_{16}^8 =$

$$\mathbf{Q}(\zeta_{16}) = \mathbf{Q} + \mathbf{Q}\zeta_{16} + \mathbf{Q}\zeta_{16}^2 +$$
$$+ \mathbf{Q}\zeta_{16}^4 + \mathbf{Q}\zeta_{16}^5 + \mathbf{Q}\zeta_{16}^6 +$$

Reasonably short basis
for the infinite-dimensional
lattice of $\mathbf{Q}(\zeta_8)^*$ logs,
shown truncated after 2 digits:

$$\begin{array}{cccccc} 1.76 & -1.76 & 0 & 0 & 0 & \ldots \\ 1.22 & -0.53 & -0.69 & 0 & 0 & \ldots \\ 1.09 & 1.09 & 0 & -2.19 & 0 & \ldots \\ 1.09 & 1.09 & 0 & 0 & -2.19 & \ldots \\ \vdots & & & & & \end{array}$$

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$\zeta_{16} = \exp(2\pi i/16)$ so $\zeta_{16}^8 = -1$.

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1.76	−1.76	0	0	0	...
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8th roots of -1 in $\mathbf{C}$:
$\zeta_{16}^{\pm 1}, \zeta_{16}^{\pm 3}, \zeta_{16}^{\pm 5}, \zeta_{16}^{\pm 7}$.

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mapping ζ_{16} to ζ_{16}^j.

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f $\mathbf{Q}(\zeta_8)^*$ logs,

runcated after 2 digits:

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.53	−0.69	0	0	...
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$\mathbf{Z}[\zeta_{16}] =$

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$\mathbf{Z}[\zeta_{16}] = \mathbf{Z} + \mathbf{Z}\zeta_{16}$
$+ \mathbf{Z}\zeta_{16}^4 + \mathbf{Z}\zeta_{16}^5$

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$\{\infty\}$-integers, meaning

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$\{\infty\}$-units: $\zeta_{16}^{\mathbf{Z}} u_1^{\mathbf{Z}} u_3^{\mathbf{Z}} u_5^{\mathbf{Z}}$ where
$u_1 = 1 + \zeta_{16} + \zeta_{16}^2$,
$u_3 = 1 + \zeta_{16}^3 + \zeta_{16}^6 = \sigma_3(u_1)$,
$u_5 = 1 + \zeta_{16}^5 + \zeta_{16}^{10} = \sigma_5(u_1)$.

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$u_1 = 1 + \zeta_{16} + \zeta_{16}^2$,

$u_3 = 1 + \zeta_{16}^3 + \zeta_{16}^6 = \sigma_3(u_1)$,

$u_5 = 1 + \zeta_{16}^5 + \zeta_{16}^{10} = \sigma_5(u_1)$.

Exercise: $u_1 u_3 u_5 u_7 = -1$ where $u_7 = 1 + \zeta_{16}^7 + \zeta_{16}^{14} = \sigma_7(u_1)$.

The 16th cyclotomic field

$\zeta_{16} = \exp(2\pi i/16)$ so $\zeta_{16}^8 = -1$.

$$\mathbf{Q}(\zeta_{16}) = \mathbf{Q} + \mathbf{Q}\zeta_{16} + \mathbf{Q}\zeta_{16}^2 + \mathbf{Q}\zeta_{16}^3 + \mathbf{Q}\zeta_{16}^4 + \mathbf{Q}\zeta_{16}^5 + \mathbf{Q}\zeta_{16}^6 + \mathbf{Q}\zeta_{16}^7.$$

8th roots of -1 in $\mathbf{C}$:
$\zeta_{16}^{\pm 1}, \zeta_{16}^{\pm 3}, \zeta_{16}^{\pm 5}, \zeta_{16}^{\pm 7}$.

Each odd integer j has a unique ring morphism $\sigma_j : \mathbf{Q}(\zeta_{16}) \to \mathbf{C}$ mapping ζ_{16} to ζ_{16}^j.

Define $|x|_{\infty_j} = |\sigma_j(x)|^2$.

Inequivalent: $\infty_1, \infty_3, \infty_5, \infty_7$.

$\{\infty\}$-integers, meaning
$\{\infty_1, \infty_3, \infty_5, \infty_7\}$-integers:

$$\mathbf{Z}[\zeta_{16}] = \mathbf{Z} + \mathbf{Z}\zeta_{16} + \mathbf{Z}\zeta_{16}^2 + \mathbf{Z}\zeta_{16}^3 + \mathbf{Z}\zeta_{16}^4 + \mathbf{Z}\zeta_{16}^5 + \mathbf{Z}\zeta_{16}^6 + \mathbf{Z}\zeta_{16}^7.$$

$\{\infty\}$-units: $\zeta_{16}^{\mathbf{Z}} u_1^{\mathbf{Z}} u_3^{\mathbf{Z}} u_5^{\mathbf{Z}}$ where
$u_1 = 1 + \zeta_{16} + \zeta_{16}^2$,
$u_3 = 1 + \zeta_{16}^3 + \zeta_{16}^6 = \sigma_3(u_1)$,
$u_5 = 1 + \zeta_{16}^5 + \zeta_{16}^{10} = \sigma_5(u_1)$.
Exercise: $u_1 u_3 u_5 u_7 = -1$ where
$u_7 = 1 + \zeta_{16}^7 + \zeta_{16}^{14} = \sigma_7(u_1)$.

Logs of u_1, u_3, u_5, truncated:

2.09	1.13	−2.89	−0.33
1.13	−0.33	2.09	−2.89
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$\zeta_{16}^4 + \mathbf{Q}\zeta_{16}^5 + \mathbf{Q}\zeta_{16}^6 + \mathbf{Q}\zeta_{16}^7$.

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1.94 –
⋮

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$_{16} + \mathbf{Q}\zeta_{16}^2 + \mathbf{Q}\zeta_{16}^3$
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$\mathbf{C}$:

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In the infinite-dime
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after the four ∞ c

2.09	1.13	−2.8
1.13	−0.33	2.0
−2.89	2.09	−0.3
1.34	1.01	0.2
1.94	−0.68	0.9
⋮		

$-1.$

$+\mathbf{Q}\zeta_{16}^3$

$\mathbf{Q}\zeta_{16}^7.$

ique

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2.09	1.13	−2.89	−0.33	
1.13	−0.33	2.09	−2.89	
−2.89	2.09	−0.33	1.13	
1.34	1.01	0.21	−1.88	−
1.94	−0.68	0.98	0.58	
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−2.89	2.09	−0.33	1.13

In the infinite-dimensional lattice of $\mathbf{Q}(\zeta_{16})^*$ logs, a diagonal starts after the four ∞ columns:

2.09	1.13	−2.89	−0.33	0	0
1.13	−0.33	2.09	−2.89	0	0
−2.89	2.09	−0.33	1.13	0	0
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The general picture: Number of ∞ columns is between $n/2$ and n for a degree-n number field, and a diagonal appears almost immediately after the ∞ columns.