# On the Multiplicative Complexity of Symmetric Boolean Functions

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# Outline

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# Boolean functions and circuits

**We focus on Boolean functions (i.e., predicates)**

- $f : \{0,1\}^n \to \{0,1\}$ with $n$ bits of input and **1** bit of output.
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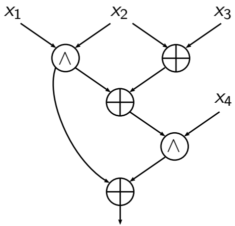
(A directed acyclic graph of gates, with inputs as sources, and with outputs as sinks.)




Example gates (fanin 2)

| input bits | output bits | |
|---|---|---|
| | AND ($\wedge$) | XOR ($\oplus$) |
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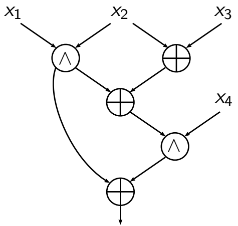


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- For nonlinear gates, we focus on AND gates with fanin 2.
- For linear gates, we focus on XOR gates with arbitrary fanin.

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## Why useful to find circuits with minimal MC?

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Notes:

- Finding the MC of a Boolean function is hard
- Almost all $f \in \mathcal{B}_n$ have MC $\geq 2^{n/2} - n - 1$; all $\leq 3 \cdot 2^{(n-1)/2} - \mathcal{O}_n$

# Symmetric Boolean functions

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Examples of classes of symmetric $n$-bit functions:

- Elementary symmetric ($\Sigma_k^n$): sum of all monomials of degree $k$ (Note: Any $f \in \mathcal{S}_n$ is a linear sum of $\Sigma_i^n$'s)
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$Maj_3$ — majority bit out of three (outputs 1 iff at least two 1s in input):
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- **Easier start-point for certain MC analyses?**
  $\mathcal{S}_n$ has $2^{n+1}$ functions; $\mathcal{B}_n$ has $2^{2^n}$ functions.
  Compared with $\mathcal{B}_n$, can we more easily characterize MC for $\mathcal{S}_n$?

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- Devise "twin" technique to analyze MC of symmetric functions
- Answer two open questions: $c_\wedge(\Sigma_4^8) = 6$; $c_\wedge(E_4^8) = 6$
- Characterize MC of functions in $\mathcal{S}_n$, for up to $n = 10$ variables:
  $n \in \{7, 8, 9, 10\} \wedge f \in \mathcal{B}_n \Rightarrow c_\wedge(f) \leq n - 1$

# Outline

# Affine equivalence

**Affine equivalence class.** $f$ and $g$ (from $\mathcal{B}_n$) are affine equivalent ($f \sim g$) if

$$f(x) = g(Ax + a) + b \cdot x + c, \text{ where:}$$

- $A$ is a non-singular $n \times n$ matrix over $\mathbb{F}_2$;
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**MC of equivalence class.** Multiplicative complexity is invariant under affine transformations: $f \sim g \Rightarrow c_\wedge(f) = c_\wedge(g)$

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | – | – | – | – | – | – | 1 |
| 2 | 1 | 1 | – | – | – | – | – | 2 |
| 3 | 1 | 1 | 1 | – | – | – | – | 3 |
| 4 | 1 | 1 | 3 | 3 | – | – | – | 8 |
| 5 | 1 | 1 | 3 | 17 | 26 | – | – | 48 |
| 6 | 1 | 1 | 3 | 24 | 914 | 148,483 | 931 [ÇTP18] | 150,357 [Mai91] |

Table 1: number of classes per $n$ (#vars) and $k$ (MC)

# Max MC of Boolean Functions with $n \leq 6$

- $f \in \mathcal{B}_4$ (8 classes) $\Rightarrow c_\wedge(f) \leq 3$ [TP15]
- $f \in \mathcal{B}_5$ (48 classes) $\Rightarrow c_\wedge(f) \leq 4$ [TP15]
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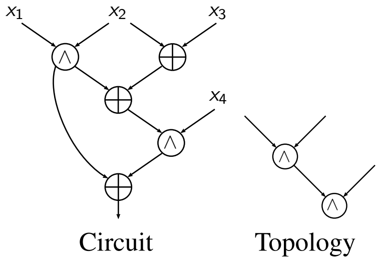
## (Circuit) Topologies [CCFS15]

E.g.: $f = x_1x_2x_3 + x_1x_2 + x_1x_4 + x_2x_3 + x_4$



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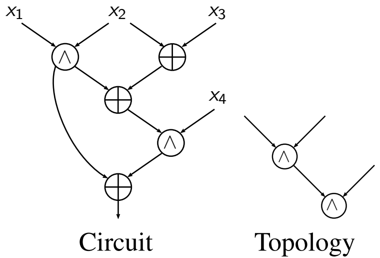


## Method [ÇTP18]

- Iterate over all topologies with 1, 2, 3, … AND gates

| # AND gates | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| # topologies | 1 | 2 | 8 | 84 | 3,170 | 475,248 |

- For each topology, mark the classes generated by circuits.
- Max MC for $n = 6$ is found when all classes are marked.

# Some prior results on the MC of symmetric functions

- Functions in $\mathcal{B}_n$ have circuits with $\leq n + 3\sqrt{n}$ `AND` gates [BPP00]
- The MC of an $n$-bit *nonlinear* symmetric function is at least $\lfloor \frac{n}{2} \rfloor$ [BP08]
- The MC of $\Sigma_2^n$ is $\lfloor \frac{n}{2} \rfloor$; the MC of $\Sigma_3^n$ is $\lceil \frac{n}{2} \rceil$, ... [BP08]

Table A.1 from [BP08]:
MC complexity of the elementary symm $\Sigma_i^n$

| $n \backslash i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | – | – | – | – | – |
| 4 | 2 | 2 | 3 | – | – | – | – |
| 5 | 2 | 3 | 3 | 4 | – | – | – |
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| 7 | 3 | 4 | 4 | 5 | 5 | 6 | – |
| 8 | 4 | 4 | **5–6** | 5 | 6 | 6 | 7 |

Table A.3 from [BP08]:
MC complexity of the counting function $E_i^n$

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| 3 | 2 | 2 | 2 | 2 | – | – | – | – | – |
| 4 | 3 | 2 | 2 | 2 | 3 | – | – | – | – |
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**Two concrete open questions:**

1. What is the MC of $\Sigma_4^8$? (Is it 5 or 6?)
2. What is the MC of $E_4^8$? (Is it 6 or 7?)

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# Boolean Functions with Twin Variables

(Towards facilitating the analysis of symmetric Boolean functions)

Definition (twin variables): Let $f(x) = \boldsymbol{x_i x_j} g(x) + h(x)$, where $g$ and $h$ do not depend on $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$. Then, $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$ are called **twins** in $f$.

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Example: $f(\boldsymbol{x_1}, x_2, x_3, \boldsymbol{x_4}) = \boldsymbol{x_1 x_4}(1 + x_2 + x_2 x_3) + x_3$

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Replace $x_1 x_n$ by $y_1$ and let $f'(y_1, x_2, ..., x_{n-1}) = f(x_1, x_2, ..., x_n)$.

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But what about symmetric functions ($\mathcal{S}_n$)? (next slide)

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- $f = \Sigma_2^3 = x_1x_2 \oplus x_1x_3 \oplus x_2x_3 = (x_1 \oplus x_3)(x_2 \oplus x_3) \oplus x_3$
- Var transform ($\boldsymbol{\tau}$): $x_1 \to A + C$; $x_2 \to A + B$; $x_3 \to A + B + C + 1$
- Result: $\Sigma_2^3 = (B \oplus 1)(C \oplus 1) \oplus A \oplus B \oplus C \oplus 1 = A \oplus \boldsymbol{BC}$

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Intuition:

- For any $n$ and $k$, $\boldsymbol{\tau}$ applied to $\Sigma_k^n$ combines $\boldsymbol{B}$ and $\boldsymbol{C}$ as **twins**

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Example: analysis of $f \in \mathcal{S}_8$ becomes analysis of $f' \in \mathcal{B}_{\boldsymbol{6}}$

# Multiplicative Complexity of $E_4^8$ and $\Sigma_4^8$

Using the Twin technique:

- Reduce # variables (from 8 to 6): $f \in \left\{E_4^8, \Sigma_4^8\right\} \to f' \in \mathcal{B}_6$
- Find MC-optimal circuit for $f' \in \mathcal{B}_6$
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Case $f = E_4^8$ (counting function):

- It was known that $c_\wedge(f) \in \{6, 7\}$
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Case $f = \Sigma_4^8$ (elementary symmetric function):

- It was known that $c_\wedge(f) \in \{5, 6\}$
- (Cheap) If twin-conj true: $c_\wedge(f') = 4$ directly implies $c_\wedge(f) = 4+2 = 6$
- (Expensive) No 5-AND topology can generate $f$, hence $c_\wedge(f) = 6$

# Transformation and SLPs (just for a glimpse)

Affine transformation from $f \in \mathcal{S}_8$ to $f' \in \mathcal{T}_6$:

- $(x_1, x_2, x_8) \to (x_1 \oplus x_2 \oplus x_8 \oplus 1,\ \ x_2 \oplus x_8 \oplus 1,\ \ x_1 \oplus x_2 \oplus 1)$
- $(x_3, x_4, x_7) \to (x_3 \oplus x_4 \oplus x_7 \oplus 1,\ \ x_4 \oplus x_7 \oplus 1,\ \ x_3 \oplus x_4 \oplus 1)$
- $(x_5, x_6) \to (x_5, x_6)$

SLP for $f = E_4^8$ (counting function):

$a_0 = (1 \oplus x_2 \oplus x_8) \wedge (1 \oplus x_1 \oplus x_2)$
$a_1 = (1 \oplus x_4 \oplus x_7) \wedge (1 \oplus x_3 \oplus x_4)$
$a_2 = (a_0 \oplus a_1 \oplus 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_7 \oplus x_8) \wedge (a_0)$
$a_3 = (1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_7 \oplus x_8) \wedge (1 \oplus x_1 \oplus x_2 \oplus x_5 \oplus x_8)$
$a_4 = (a_2 \oplus 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8) \wedge (a_0 \oplus a_1 \oplus a_3 \oplus 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_7 \oplus x_8)$
$a_5 = (1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8) \wedge (a_2 \oplus a_4)$
$f = a_5 \oplus 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8$

SLP for $f = \Sigma_4^8$ (elementary symmetric function):

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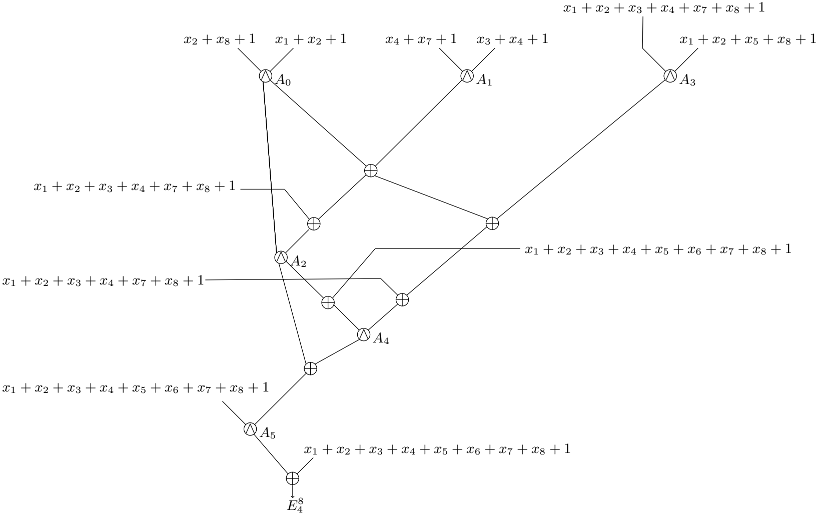
$a_5 = (a_2) \wedge (a_3)$
$f = a_0 \oplus a_4 \oplus a_5 \oplus 1 \oplus x_1 \oplus x_2 \oplus x_8$

# MC-optimal circuit for $E_4^8$ (just for a glimpse)



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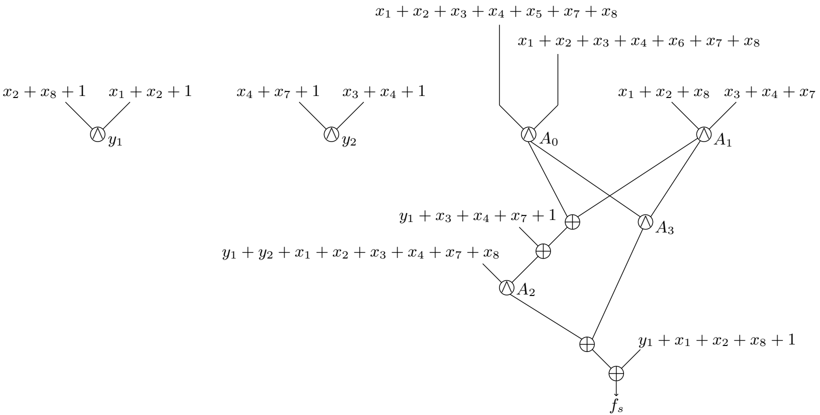

# MC of Symmetric Functions with $n \leq 10$

Prior lemma ([BP08]): $c_{\wedge}(f \in \mathcal{S}_7) \leq 8$

Using the *twin* technique and the ability to find MC for $f \in \mathcal{B}_{n \leq 6}$, we get:

$$n \in \{7, 8, 9, 10\} \Rightarrow c_{\wedge}(f \in \mathcal{S}_n) \leq n - 1$$

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|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total | $*$ |
| 1 | 4 | | | | | | | | | | 4 | |
| 2 | 4 | 4 | | | | | | | | | 8 | |
| 3 | 4 | 4 | 8 | | | | | | | | 16 | |
| 4 | 4 | | 12 | 16 | | | | | | | 32 | ✓ |
| 5 | 4 | | 4 | 24 | 32 | | | | | | 64 | |
| 6 | 4 | | | 12 | 48 | 64 | | | | | 128 | |
| 7 | 4 | | | 4 | 16 | 104 | 128 | | | | 256 | Twin |
| 8 | 4 | | | | 12 | 16 | 224 | 256 | | | 512 | conj. |
| 9 | 4 | | | | 4 | 8 | 48 | 448 | 512 | | 1024 | (TC) |
| 10 | 4 | | | | | 12 | 0 | 96 | 712 | 1224 | 2048 | |

Legend: $n$ (# input vars); $k$ (# `AND` gates); TC (twin conjecture)

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| 2 | 4 | 4 | | | | | | | | | 8 | |
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Note: all cells are multiple of 4, since MC is independent of sum by $\Sigma_0^n$ and $\Sigma_1^n$

# Outline

# Summary and further research

**Summary**

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- Devised the twin method for reducing # variables
- Answered two open questions: $c_\wedge(\Sigma^8_4) = 6$; $c_\wedge(E^8_4) = 6$
- Gave upper bounds (conjectured tight) for up to $n = 10$ variables
- (Not shown here) new non-tight upper-bounds for higher $n$

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Thank you for your attention!

# References


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