On the Multiplicative Complexity of 6-variable Boolean Functions

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What is Multiplicative Complexity?

Multiplicative complexity is a complexity measure that is defined as the minimum number of AND gates required to implement a function f by a circuit over the basis (AND, XOR, NOT).

Why do we count the AND gates?

- **Lightweight Cryptography:** Efficient implementations needed for resource-constrained devices (e.g. RFID tags). The technique of minimizing the number of AND gates, and then optimizing the linear components leads to the implementations with low gate complexity.
- **Secure multi-party computation:** Reducing the number of AND gates improves the efficiency of secure multi-party protocols (e.g. conducting online auctions in a way that the winning bid can be determined without opening the losing bids).
- **Side channel attacks:** Minimizing the number of AND gates is necessary when implementing a masking scheme to prevent side-channel attacks.
- **Cryptanalysis of cryptographic primitives:** Primitives with low multiplicative complexity may be susceptible to algebraic cryptanalysis.

Some Properties of Multiplicative Complexity

- Multiplicative complexity of a function with degree d is at least $d-1$.
- Multiplicative complexity is invariant w.r.t affine transformation.
  - f and g are affine equivalent, if there exists an affine transformation of the form $f(x) = g(Ax + a) + b \cdot x + c$, where A is a non-singular $n \times n$ matrix over $\mathbb{F}_2$; x, a are column vectors over $\mathbb{F}_2$; b is a row vector over $\mathbb{F}_2$.
  - If f and g are affine equivalent, they are said to be in the same equivalence class and they have the same multiplicative complexity.
- Multiplicative complexity of a randomly selected n-bit Boolean function is at least $2^{n/2} - \mathcal{O}(n)$. No specific n-bit Boolean function has been proven to have multiplicative complexity larger than $n-1$ for any n.

4- and 5-bit Boolean Functions (Turan and Peralta, 2014)

Turan and Peralta (2014) showed that multiplicative complexity is

- ≤ 3 for $f \in B_4$ (8 equivalence classes),
- ≤ 4 for $f \in B_5$ (48 equivalence classes).

Method

1. Find a simple representative from each equivalence class.
2. Find a circuit with small number of AND gates.
3. Check if it is optimal using the degree bound.

Equivalence classes for $n = 4$

Class	Representative
1	x_1
2	x_1x_2
3	$x_1x_2 + x_3x_4$
4	$x_1x_2x_3$
5	$x_1x_2x_3 + x_1x_4$
6	$x_1x_2x_3x_4$
7	$x_1x_2x_3x_4 + x_1x_2$
8	$x_1x_2x_3x_4 + x_1x_2 + x_3x_4$

6-bit Boolean Functions

The approach of Turan & Peralta does not work for $n = 6$, since

- The number of equivalence classes is 150 537, and
- Simple heuristics do not find optimal circuits, as representatives are more complex.
- For some classes, it is not possible to verify optimality using the degree bound.

Our approach

Exhaustively construct all Boolean circuits with 1,2, 3, . . . AND gates, and mark the Boolean functions that can be generated by the circuits until all 6-bit Boolean functions are generated.

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Exhaustively construct all Boolean ~~circuits~~ topologies with 1,2, 3, ... AND gates, and mark the Boolean functions that can be generated by the circuits until a function from each equivalence class is generated.

Boolean circuit and Topology of a circuit (Codish et al, 2015)

Definition (Boolean circuit)

For a given $n \in \mathbb{N}$, let $X_n = \{x_1, x_2, \ldots, x_n\}$ denote the n inputs to a circuit. A Boolean circuit C with n inputs and k AND gates is a pair $\mathcal{C} = (\mathcal{A}, \mathcal{O})$, where:

- $\mathcal{A} = \{a_1, \ldots, a_k\}$ is a list of k AND gates, where the i-th AND gate inputs L_i and R_i with $L_i, R_i \in \langle 1, x_1, \ldots, x_n, L_1.R_1, \ldots, L_{i-1}.R_{i-1} \rangle$.
- $\mathcal{O} \in \langle 1, x_1, \ldots, x_n, L_1.R_1, \ldots, L_k.R_k \rangle$ is the output gate.

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Definition (Topology)

A topology of a circuit $C = (\mathcal{A}, \mathcal{O})$ is the set of AND gates $\mathcal{A}$, except that $L \cup R \subset \mathcal{A}$ for all $\langle L, R\rangle \in \mathcal{A}$. Given an AND-XOR circuit $C = \langle \mathcal{A}, \mathcal{O}\rangle$, the topology of $\mathcal{C}$ is $\langle \langle L \cap \mathcal{A}, R \cap \mathcal{A}\rangle \,|\, \langle L, R\rangle \in \mathcal{A}\rangle$.

Example: Boolean Circuit and Topology

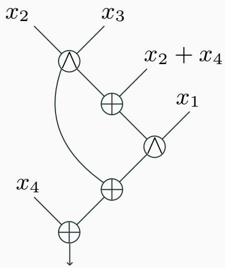
Let $f = x_1x_2x_3 + x_1x_2 + x_1x_4 + x_2x_3 + x_4$.

The circuit $C = \langle \mathcal{A}, \mathcal{O} \rangle$ is represented as $\mathcal{A} = \langle a_1, a_2 \rangle$

$a_1 = \langle \{x_2\}, \{x_3\} \rangle$

$a_2 = \langle \{a_1, x_2, x_4\}, \{x_1\} \rangle$

$\mathcal{O} = \langle \{x_4\}, \{a_1, a_2\} \rangle$



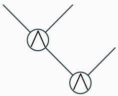
The topology of C is represented as

$\mathcal{A} = \langle a_1, a_2 \rangle$

$a_1 = \langle \emptyset, \emptyset \rangle$

$a_2 = \langle \{a_1\}, \emptyset \rangle$

$\mathcal{O} = \langle \emptyset, \{a_1, a_2\} \rangle$



Constructing Circuit Topologies

Let T_k be the set of all topologies with k AND gates. We use an iterative method to construct T_{k+1} as follows:

1. Let S be an empty set.
2. For each topology $t \in T_k$,
    2.1 For all choices of (L_{k+1}, R_{k+1}) (L_{k+1} and R_{k+1} can take on all 2^k possible combinations of previous k AND gates),
        2.1.1 Let t' be a new topology constructed by adding a new AND gate a_{k+1} with inputs (L_{k+1}, R_{k+1}) to t.
        2.1.2 $S = S \cup t'$
3. We eliminate redundant topologies (due to symmetry). $T_{k+1} = S$.

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Number of topologies for k up to 6

k	1	2	3	4	5	6
$\|T_k\|$	1	2	8	84	3 170	475 248

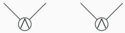
Constructing Circuit Topologies

Topologies with 1 AND gate

Constructing Circuit Topologies

Topologies with 1 AND gate

Topologies with 2 AND gates



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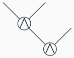
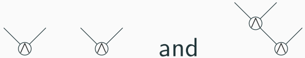


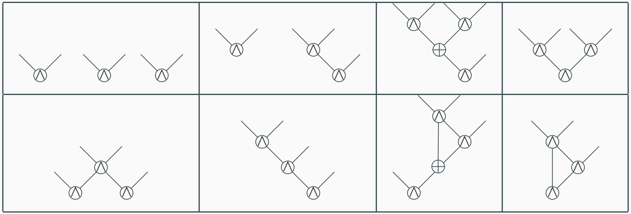
Constructing Circuit Topologies

Topologies with 1 AND gate

Topologies with 2 AND gates

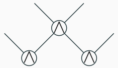


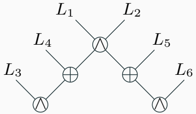
Topologies with 3 AND gates



Evaluating Topologies to Generate Boolean Functions

- A topology with k AND gates can be supplied $2k$ linear function inputs $X = (L_1, \ldots, L_{2k})$. Trying all inputs becomes quickly infeasible since there are 2^{2kn} choices (2^{60} inputs for $n = 6,\ k = 5$).
- Any affine transformation of the inputs $A(X) = (A(L_1), \ldots, A(L_{2k}))$ will produce a function from the same equivalence class. Hence, the inputs that are affine transformations of each other need not be considered.
- The number of inputs corresponds to the Gaussian binomial coefficient $\binom{2k}{n}_2$ ($\approx 2^{26}$ inputs for $n = 6,\ k = 5$).





Computation Summary

- Generated all topologies ≤ 6 AND gates.
- For each topology having $k = 1, 2, 3, 4, 5$ AND gates, all equivalence classes each topology can produce is found.
- 149 426 equivalence classes out of 150 357 generated with at most 5 AND gates.
- Remaining 931 equivalence classes were generated from a selection of 6 AND gate topologies.
- Computations were done on a cluster (Intel Xeon E5-2630 processor, 64GB RAM) and took 38 422 core hours.

Multiplicative Complexity Distribution for $n = 6$

Multiplicative complexity distribution of the equivalence classes and functions for $n = 6$

MC	#classes	#functions	$\log_2(\#functions)$
0	1	128	7.00
1	1	83 328	16.34
2	3	73 757 184	26.13
3	24	281 721 079 808	38.03
4	914	7 944 756 861 878 272	52.81
5	148 483	18 344 082 080 963 133 440	63.99
6	931	94 716 954 089 619 456	56.39

Conclusion

- Multiplicative complexity distribution of 6-bit Boolean functions is found.
- Showed that the multiplicative complexity is ≤ 6 for $f \in B_6$.
- Showed that there exists $f \in B_6$ with multiplicative complexity 6, e.g.,
  - A function with 6 monomials: $x_1x_5 + x_3x_6 + x_3x_4x_5 + x_2x_4 + x_1x_2x_6 + x_1x_2x_3x_4x_5x_6$
  - A function with algebraic degree 4: $x_4x_5 + x_3x_4x_5 + x_2x_5 + x_2x_4 + x_2x_4x_6 + x_1x_5x_6 + x_1x_4 + x_1x_3 + x_1x_2x_4x_5 + x_1x_2x_3x_6$

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Thanks!